Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 19 December 2016, 09:00–11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 5 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (22 points)

Consider the function

$$f(z) = u(x,y) + iv(x,y) = e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y).$$

(a) (12 points) Prove that $f(z)$ is entire using the Cauchy-Riemann equations *Hint: the Cauchy-Riemann equations do not allow by themselves to claim that the function is* analytic*; more conditions must be satisfied and they should be part of your answer.*

(b) (10 points) Write $f(z)$ as a function of z (instead of x and y separately).

Question 2 (18 points)

The principal value of arctan is defined as

$$\operatorname{Arctan}(z) = \frac{i}{2}\operatorname{Log}\frac{i+z}{i-z}.$$

(a) (6 points) Compute $\operatorname{Arctan}(-1)$ using the definition of $\operatorname{Arctan}(z)$.

(b) (12 points) Show that for $x \in \mathbb{R}$ we have $\operatorname{Arctan}(x) \in (-\pi/2, \pi/2)$. *Hint: show that for $x \in \mathbb{R}$ we have $\left|\frac{i+x}{i-x}\right| = 1$.*

Question 3 (18 points)

Show that

$$\left|\int_C \frac{e^z}{\bar{z}+2}dz\right| \leq \pi e^2,$$

where C is the positively oriented circle $|z-1| = 1$.

Question 4 (18 points)

A function $f(z)$ is analytic in a domain D. Prove that if the modulus $|f(z)|$ is constant in D then the function $f(z)$ is constant in D.

Question 5 (14 points)

Compute the value of the integral

$$\int_\Gamma \frac{\cos(\pi z)}{(z-1)(z-3)^2}\, dz$$

where Γ is the closed contour shown in Figure 1.

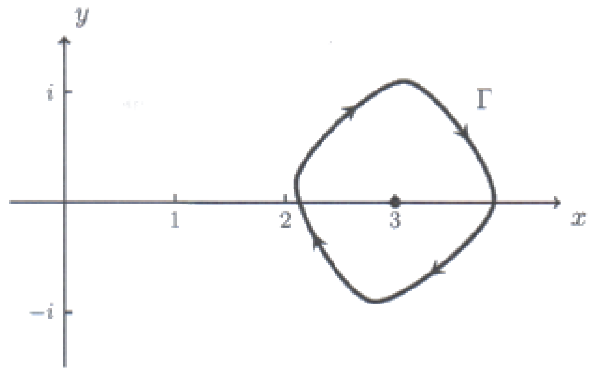

Figure 1: Contour Γ for Question 5.